Published by Institute of Physics Publishing for SISSA/ISAS
Received: *July 21, 2003*
Accepted: *September 16, 2003*

Dyonic instanton as supertube between D4-branes

Seok Kim

School of Physics, Seoul National University
Seoul 151-747, Korea
E-mail: calaf2@snu.ac.kr

Kimyeong Lee

Korea Institute for Advanced Study
Seoul 130-722, Korea
E-mail: klee@kias.re.kr

Abstract: We study dyonic instantons in (4+1) dimensional Yang-Mills theory. Especially we consider the most general two instanton solution given by the Jackiw-Nohl-Rebbi ansatz and find its dyonic version. By exploring the zeros of the Higgs field, we rederive the porism structure of triangles in this solution and also find the magnetic monopole string loop. This leads to the identification of dyonic instanton with the supertube inserted between D4-branes.

Keywords: D-branes, Brane Dynamics in Gauge Theories, Solitons Monopoles and Instantons, Extended Supersymmetry.

Contents

1. Introduction

The gauge field configurations with self-dual field strength in four dimensions are called instantons. In 4+1 dimensional spacetime, instantons appear as solitons in the theory. Classical instantons have zero modes some of which describe the sizes of instantons. When the gauge symmetry is spontaneously broken, instantons collapse unless they carry nonzero electric charge and become dyonic instantons [1]–[3]. In type-IIA string theory the low energy dynamics of parallel D4-branes lying close to each other is described by 4+1 dimensional Yang-Mills theory with 16 supersymmetries, while BPS D0-branes on D4-branes are realized as BPS instantons in this theory. As fundamental strings connecting parallel branes are electrically charged particles in the broken phase, dyonic instantons would be considered as BPS composite objects made of D0-branes and fundamental strings connecting D4-branes. Refs. [4, 5] also treat various aspects of dyonic instantons or calorons.

On the other hand, there has been some study of supertubes which are 1/4 BPS objects made of fundamental strings, D0-branes and D2-branes in type-IIA string theory [6]. They are 2-dimensional tube-like object with arbitrary shape of cross section [7]. It has been shown that supertubes can end on a single D4-brane, while remaining BPS [8]. Supertubes can also be inserted between two parallel D4-branes, remaining BPS. A BPS configuration with infinite number of dyonic instantons and parallel magnetic monopole and anti-magnetic monopole strings has been found [9], which can be interpreted as a supertube made of parallel D2 anti-D2-branes. One would expect that general dyonic instantons should be interpreted as supertubes inserted between D4-branes.

However dyonic instantons of 't Hooft type studied in ref. [1, 3] do not show magnetic monopole string, or tube-like structure. These solutions seem to describe collapsed tubes. In this paper we explore the most general two instanton solution in the SU(2) gauge theory, which is given by the Jackiw-Norl-Rebbi ansatz. We find the solution of the covariant Laplace equation for the adjoint scalar field, which leads to the BPS dyonic solutions. The Higgs field configuration shows how two D4-branes are deformed near the dyonic instanton. Especially, the set of points where the Higgs field vanishes is shown to be a closed magnetic monopole loop, revealing a tube-like structure connecting two D4-branes. This allows us to interpret the dyonic instanton as a supertube interpolating two D4-branes.

Supertube, which is a tube-like 1/4 BPS configuration with a translation symmetry along the tube, have been studied initially in the DBI action of D2-branes in type-IIA theory [6]. Later, they have been studied in the matrix theory [11] and in the supergravity [12]. Various configurations like parallel cylindrical tubes [10] and D2 anti-D2 pair [11] have been found, while it was also shown that the cross section of a supertube can take arbitrary shape [7]. In energetics the supertube energy can be assigned to those of fundamental strings and D0-branes, while the bound energy of D0 and fundamental strings to D2-branes cancel the energy of D2-branes. Hence D2-brane cross section can take arbitrary shape, maintaining BPS condition. In addition supertubes can end on D4-branes, reducing the supersymmetry to 1/8 of the superstring theory, and so carrying 4 supersymmetries. Supertubes can also be inserted between D4-branes. The obvious field theoretic interpretation seems to be dyonic instantons in 4+1 dimensional Yang-Mills theory on D4-branes. This analysis can be done in the DBI action.

The low energy dynamics of two closely lying D4-branes is described by SU(2) gauge theory with 16 supersymmetries. Supertubes inserted between D4-branes are expected to be represented as dyonic instantons in the corresponding field theory. The Higgs field describes the profile of the deformation of D4-branes near dyonic instantons. The shape of supertubes between D4-branes should appear as the zero points of the Higgs field, where two D4-branes meet. Previous work on dyonic instantons based on the 't Hooft ansatz has shown that D4-branes meet on isolated points, instead of some loop. However the 't Hooft type solutions are not the most general instanton solution. The tube-like structure may appear in the more general dyonic instanton solutions.

In this work, we exlore the most general two instanton solution given by the Jackiw-Nohl-Rebbi (JNR) ansatz [13]. The JNR ansatz for two instantons are characterized by three scale parameters and three positions in four dimensional space. These threes-some quantities are related to each other by the local gauge transformation. Especially the triangle connecting three positions defines a circle on which three points lie and an ellipse which three sides of the triangle touches tangentially [14]. Under a local gauge transformation, the circle and the ellipse remain invariant, while the triangle changes on circle. This local gauge transformation is characterized by a single parameter and this one parameter family of triangle is called a porism of triangles. As the magnetic monopole string we find here is not the ellipse which appears in the porism, the physical meaning of the ellipse remains obscure.

Our method for analysis is the ADHM construction of instantons [15]. The ADHM method for the 't Hooft solutions has been known. We find its generalization for the JNR solutions, which is new as far as we know. Then we use the ADHM method to solve the covariant Laplace equation for the adjoint scalar field. The ADHM version of the Laplace equation is a matrix equation which can be solved easily in the two instanton case [16].

JHEP09(2003)035

In section 2, we summarize the known physics of dyonic instantons in the spontaneously broken SU(2) gauge theory. We argue that magnetic monopole string appears naturally along the zeros of the Higgs field. In section 3, we summarize the ADHM method and introduce the ADHM data for the JNR instanton solutions. We set up the matrix equation for the solution of the covariant Laplace equation. In section 4, we find the Higgs field solution explicitly. We rederive the porism of the parameters and find magnetic monopole strings. Also we investigate the 't Hooft solution limit to see explicitly that the collapsed tube interpretation of 't Hooft solution is correct. We conclude with some remarks in section 5.

2. 't Hooft and Jackiw-Nohl-Rebbi instantons

We start with the 5-dimensional 16 supersymmetric Yang-Mills theory with SU(N) gauge group, which describes the low energy dynamics of N parallel D4-branes whose distances between them are smaller than the string scale. The lagrangian is given in the standard form. The supersymmetry implies a Bogomolnyi bound on the energy functional. With only a single scalar field being excited, the bosonic part of the energy functional can be reexpressed as

$$\mathcal{E} = \frac{1}{e^2}\int d^4x \; \mathrm{tr}\left[\frac{1}{4}(F_{\mu\nu} - \tilde{F}_{\mu\nu})^2 + (E_\mu - D_\mu\phi)^2 + (D_0\phi)^2\right] + \frac{8\pi^2}{e^2}\kappa + \frac{2}{e^2}Q_e\,, \tag{2.1}$$

where $\mu,\nu = 1,2,3,4$ and $E_\mu = F_{\mu 0}$. The quantities

$$\kappa = \frac{1}{16\pi^2}\int d^4x \; \mathrm{tr}F_{\mu\nu}\tilde{F}_{\mu\nu}, \;\; Q_e = \int d^4x \; \partial_\mu \mathrm{tr}E_\mu\phi\,, \tag{2.2}$$

are the instanton number and the energy due to the electric charge, respectively. The above expression leads to the energy bound when $\kappa \geq 0$ and $Q_e \geq 0$. With the change of signs in the energy expression one can give the bound on the energy for other cases. The energy bound is saturated by the configuration satisfying

$$F_{\mu\nu} = \tilde{F}_{\mu\nu}\,, \qquad E_\mu = D_\mu\phi\,, \qquad D_0\phi = 0\,. \tag{2.3}$$

The first equation is the self-dual equation for the field strength, whose solutions are instantons. It is characterized by the instanton number κ. When the gauge group is SU(2), it is known that a general κ instanton solution has 8κ zero modes modulo local gauge transformations. Thus a κ instanton solution is characterized by 8κ moduli parameters, among which three will be the global gauge transformation for the gauge group SU(2).

The field configurations should satisfy the Gauss law constraint

$$D_\mu E_\mu + i[\phi, D_0\phi] = 0\,. \tag{2.4}$$

The last two equations of eq. (2.3) with the Gauss law constraint can be combined to be

$$D_\mu D_\mu\phi = 0\,. \tag{2.5}$$

JHEP09(2003)035

(For the BPS configurations satisfying eqs. (2.3) and (2.4), we can choose the gauge where the fields are time-independent. In this gauge $A_0 = \phi$ for the BPS case.) The above equation is the covariant Laplace equation for adjoint scalar field in the background of the instanton solution. For this configuration, the electric charge contribution to the energy becomes

$$Q_e = \int_{S^3_\infty} dS^\mu \ \mathrm{tr}\,(\phi\partial_\mu\phi) = \lim_{x\to\infty} 2\pi^2 x^2 x_\mu \mathrm{tr}(\phi\partial_\mu\phi)\,. \tag{2.6}$$

The simplest type of self-dual solutions is obtained with the 't Hooft ansatz

$$A_\mu(x) = \frac{i}{2}\sigma^a\bar\eta^a_{\mu\nu}\partial_\nu \log H(x)\,, \tag{2.7}$$

where $\bar\eta^a_{\mu\nu}$ is the anti-self-dual 't Hooft tensor. The above ansatz leads to the self-dual field strength if the field $H(x)$ satisfies $(\partial^2_\mu H)/H = 0$. For the 't Hooft-type solutions

$$H(x) \equiv 1 + \sum_{i=1}^{\kappa} \frac{s_i}{|x-a_i|^2}\,. \tag{2.8}$$

This solution describes κ instantons with 5κ parameters besides three global gauge parameters.

In the background of the 't Hooft instanton solution, the covariant Laplace equation has the solution

$$\phi(x) = \frac{q}{H(x)}\,, \tag{2.9}$$

which has the asymptotic behaviour $\phi(x)\to q$ as $|x|\to\infty$ [1, 3]. The electric charge energy for this configuration is

$$Q_e = 2\pi^2 s_\Sigma\ \mathrm{tr} q^2\,, \tag{2.10}$$

where $s_\Sigma = \sum_{i=1}^{\kappa} s_i$. This solution ϕ is intrinsically abelian and has zeros at κ points $x = a_1, \ldots, a_k$. The ϕ field describes the deformation of D4-branes along a transverse direction. Asymptotically two D4-branes are separated from each other by $\sqrt{\mathrm{tr} q^2}$ in the string scale. They come together at isolated points. There is no indication of supertubes of finite size connecting two D4-branes. The 't Hooft solutions seem to describe collapsed supertubes connecting D4-branes.

Apparently there exists another type of the solutions for the above ansatz (2.7), the so-called Jackiw-Nohl-Rebbi solution with

$$H(x) \equiv \sum_{i=0}^{\kappa} \frac{s_i}{|x-a_i|^2}\,. \tag{2.11}$$

This solution describes a configuration of κ instantons. After taking out the overall constant scale of H, the number of parameters of this solution seems naively $5\kappa+4$, besides the three global gauge transformation parameters. When the points a_i lie on a single circle or line, an additional constraint due to a local gauge transformation appears, making one less parameters to be independent. For the $\kappa = 2$ case, the solution is characterized by three positions a_i in the space and they always lie on a circle. Thus, the JNR solution with

$\kappa = 2$ has 13 instead of 14 independent parameters. With the additional three parameters for the global gauge parameters, this JNR solution has 16 parameters, which is exactly the number of zero modes for the general two instantons.

When we consider the dyonic version, the solution (2.9) for the covariant scalar equation does not have good asymptotic value as it diverges. Thus, one needs to find more general solution. The key task in this work is to find and explore the solution with the right boundary condition.

There are several aspects of dyonic instantons which are generic. As the gauge symmetry is spontaneously broken from SU(2) to U(1), one can generalize the 't Hooft U(1) field strength to 4+1 dimension:

$$G_{MN} = \hat{\phi}^a F^a_{MN} - \epsilon^{abc}\hat{\phi}^a (D_M\hat{\phi})^b (D_N\hat{\phi})^c \,. \tag{2.12}$$

This becomes $G_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$ for a constant $\hat{\phi}^a = \delta^{a3}$. One can define the U(1) electric charge current j^μ by

$$j^M = \partial_N G^{NM} \,, \tag{2.13}$$

which is conserved manifestly. It is known that one can define the magnetic string current m_{MN} as

$$m^{MN} \equiv \frac{1}{2}\epsilon^{MNPQR}\partial_P G_{QR} \,, \tag{2.14}$$

by generalizing the magnetic monopole current in 3+1 dimensions. It is manifestly conserved. On the other hand one can define the covariant topological string current

$$k^{MN} = -\frac{1}{8\pi}\epsilon^{MNPQR}\epsilon^{abc} D_P\hat{\phi}^a D_Q\hat{\phi}^b D_R\hat{\phi}^c \,, \tag{2.15}$$

which is related to the magnetic monopole current by the relation $k^{MN} = m^{MN}/4\pi$. The topological current vanishes everywhere except at the zeros of ϕ. Unless the zeros of the Higgs field is highly degenerated, the topological current would not vanish. As we will see, the zeros of the Higgs field in instanton background is generically a closed loop and so can be interpreted naturally as a magnetic monopole string.

Another interesting quantity is the angular momentum which is

$$L_{\mu\nu} = \int d^4x (x_\mu T^0_{\ \nu} - x_\nu T^0_{\ \mu}) \,, \tag{2.16}$$

where T^μ_ν is the conserved energy momentum tensor. For 't Hooft type self-dual dyonic instantons the angular momentum simplifies to be anti-selfdual. There is no indication of such a simplication of the angular momentum for more general dyonic instanton configurations we study here.

3. The ADHM method

To obtain the scalar solution in the JNR background, we employ the ADHM method. As we will see, the answer is quite nontrivial and could not be reached easily by an ansatz. Let us start with a brief review of the ADHM method. For the κ-instanton configurations in the SU(2) gauge theory, we start with κ dimensional row vectors Λ_μ and $\kappa \times \kappa$ hermitean

JHEP09(2003)035

matrices Ω_μ, and introduce a quaternionic $(\kappa+1)\times\kappa$ matrix

$$\Delta(x) = \begin{pmatrix} \Lambda_\mu e_\mu \\ (\Omega_\mu - x_\mu) e_\mu \end{pmatrix}, \tag{3.1}$$

where $e_\mu \equiv (i\vec{\sigma},\ 1)$. The x dependence of $\Delta(x)$ is shown explicitly in the above equation. Note that $e_\mu \bar{e}_\nu = \delta_{\mu\nu} + i\eta^a_{\mu\nu}\sigma^a$ with the selfdual 'tHooft tensor $\eta^a_{\mu\nu}$. We require that $\Delta^\dagger\Delta$ is proportional to the identity element of quaternion and invertible. This leads to three quadratic matrix equations on Λ_μ and Ω_μ, and we call those satisfying this condition to be the ADHM data. Then we find a $\kappa+1$ dimensional quaternionic column vector $v(x)$ such that $\Delta^\dagger(x)v(x) = 0$ and $v^\dagger v = 1$, which leads to the selfdual instanton configuration $A_\mu = v^\dagger \partial_\mu v$. There is a U$(\kappa)$ group action on the ADHM data, whose parameters do not appear in the instantons. When one counts the number of free parameters which determine the instanton configuration, one finds that 8κ is the number of zero modes for κ instanton configuration in the SU(2) gauge group case.

To find the solution of the covariant Laplace equation in the adjoint representation, we start with an ansatz as [16, 1]

$$\phi = v^\dagger \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} v\,, \tag{3.2}$$

where q is the asymptotic Higgs value and also an element of the SU(2) algebra. The $\kappa\times\kappa$ matrix p belongs the identity element of quaternion and satisfies the constraint,

$$-[\Omega_\mu, [\Omega_\mu, p]] - \left\{\Lambda^\dagger_\mu \Lambda_\mu, p\right\} - 2i\eta^a_{\mu\nu} q^a \Lambda^\dagger_\mu \Lambda_\nu = 0\,, \tag{3.3}$$

where $q = q^a\sigma^a$. Here we choose the ϕ to be a hermitean matrix and so are q and p.

The $\kappa = 2$ ADHM data has been constructed a long ago [17]. But it does not lead to the JNR ansatz. Here we introduce the ADHM data for the JNR instanton solutions for κ instantons, because geometric interpretation of the data would be easier. This is achieved by mimicking the ADHM construction of the 't Hooft solution. With the quaternion position operators $y_i = (x - a_i)_\mu e_\mu$, the relavant ADHM data is

$$\Delta(x) = \begin{pmatrix} y_0 \Lambda \\ -Y_{\kappa\times\kappa} \end{pmatrix}, \tag{3.4}$$

where Λ is a row vector with real number components $(\lambda_1/\lambda_0, \ldots, \lambda_\kappa/\lambda_0)$ and Y is a quaternionic matrix $Y = \mathrm{diag}(y_1, \ldots, y_k)$. (Taking the limit $a_{04} = \lambda_0 \to \infty$, we recover the ADHM data for the 't Hooft solution.) This indeed satisfies the ADHM constraint equations, and the $(\kappa+1)$ column vector $v(x)$ is chosen such that the i-th entry is

$$v_i = \frac{\lambda_i y_i}{y_i^2} \frac{1}{\sqrt{H}}\,, \tag{3.5}$$

where $y_i^2 = |y_i|^2 = \sum_\mu y_{i\mu} y_{i\mu}$. The corresponding gauge field $A_\mu = \bar{v}\partial_\mu v$ is the JNR solution with $s_i = \lambda_i^2$. Note that the 'permutation symmetry' between (y_0, s_0), (y_1, s_1), and (y_2, s_2) is obscure in the ADHM data (3.4) but manifest in the gauge field.

To get contact with the standard form (3.1), we need to perform a transformation $\Delta \to U\Delta K$ with our JNR data (3.4), which carries same information provided $U \in U(2\kappa+2)$ and $K \in GL(\kappa, C)$. To change the matrix $\Delta(x)$ into the standard form, we choose K and U as

$$K^{-2} = 1 + \Lambda^T\Lambda\,, \qquad U = \begin{pmatrix} u & u\Lambda \\ -K\Lambda^T & K \end{pmatrix}, \tag{3.6}$$

where $u^{-2} = 1 + \Lambda\Lambda^T$. Defining the relative instanton position matrix $\mathcal{A} = \mathrm{diag}(a_1 - a_0, \ldots, a_\kappa - a_0)$, the result is

$$U\Delta(x)K = \begin{pmatrix} u\Lambda\mathcal{A}K \\ K\mathcal{A}K + a_0 - x \end{pmatrix}. \tag{3.7}$$

Note that the above expression is given in the standard form (3.1) As we make above transformation on Δ, $v(x)$ in eq. (3.5) also changes into $Uv(x)$.

It is difficult to solve the constraint (3.3) for p with ADHM data (3.7). If the solution p is known, the Higgs solution (3.2) becomes

$$\phi(x) = \frac{1}{s_\Sigma H(x)}\left[\bar{Z}qZ + Q(x)\right], \tag{3.8}$$

where $s_\Sigma = \sum_{i=0}^{\kappa} s_i$,

$$Z(x) = Z_\mu e_\mu = \sum_{i=0}^{\kappa} \frac{s_i y_{i\mu} e_\mu}{y_i^2}, \tag{3.9}$$

and

$$Q(x) = \lambda_0^2 \Lambda^T \left\{\frac{1}{Y} - \frac{1}{y_0}\right\} K\,p\,K \left\{\frac{1}{\bar{Y}} - \frac{1}{\bar{y}_0}\right\}\Lambda\,. \tag{3.10}$$

The asymptotic value is $\phi = \bar{U}_0(x)qU_0(x)$ with $U_0(x) = e_\mu x_\mu/|x|$. The permutation symmetry in $Q(x)$ would be manifest once we solve the equation (3.3) for p.

Before embarking upon the two instanton case, let us check whether the above solution for the scalar field works for a single instanton. The JNR solution for a single instanton should be gauge equivalent to that of 't Hooft type. Equally the scalar field in the above case should be gauge equivalent to the known solution of the 't Hooft type. We first note that for a single instanton with $\kappa = 1$,

$$H = \frac{s_0}{y_0^2} + \frac{s_1}{y_1^2} = \frac{(s_0+s_1)}{y_0^2 y_1^2}(x - a_x)^2 H_c\,, \tag{3.11}$$

where

$$H_c = 1 + \frac{s_c}{(x-a_c)^2}, \tag{3.12}$$

and $s_c = s_0 s_1 |a_1 - a_0|^2/(s_0+s_1)^2$, $a_c = (s_1 a_0 + s_0 a_1)/(s_0+s_1)$. Also p in eq. (3.3) vanishes identically for $\kappa = 1$. We define a global gauge transformation

$$V = \frac{(a_1 - a_0)_\mu e_\mu}{|a_1 - a_0|}, \tag{3.13}$$

and a local gauge transformation

$$U(x) = \frac{Z_\mu e_\mu}{|Z|} \tag{3.14}$$

with $Z_\mu = s_0 y_{0\mu}/y_0^2 + s_1 y_{1\mu}/y_1^2$. One can show that the gauge field and the adjoint scalar field in the JNR type with $\kappa = 1$ can be rewritten as

$$\begin{aligned} A_\mu &= \bar{U}V\left(\frac{i}{2}\bar{\eta}^a_{\mu\nu}\sigma^a\partial_\nu \ln H_c\right)\bar{V}U + \bar{U}V\partial_\mu(\bar{V}U)\,, \\ \phi &= \bar{U}V\left(\frac{\bar{V}qV}{H_c(x)}\right)\bar{V}U\,. \end{aligned} \tag{3.15}$$

This is a local gauge transformation of the 't Hooft type single instanton solution. The instanton position and scale are given as $a_{c\mu}$ and s_c. The asymptotic value of the scalar field is gauge equivalent to a constant matrix $\bar{V}qV$.

4. Two instanton solutions

Now we analyse two equations (3.3) and (3.8) for $\kappa = 2$. Without loss of generality, we align the instanton points a_0, a_1, a_2 on the 3-4 plane, i.e., $a_{i1} = a_{i2} = 0$. Then $\Omega_\mu = K\mathcal{A}_\mu K + a_{0\mu}$ appearing in eq. (3.3) is zero unless $\mu = 3, 4$, which means that the left hand side of eq. (3.3) is proportional to ${\eta_{34}}^a = \delta_{a3}$. Therefore, the matrix p is proportional to q^3, and does not depend on q^1, q^2. (With the ansatz (2.7), one has implicitly chosen a global gauge such that this plane is related to third direction in internal space.)

In order to solve eq. (3.3), we write down an explicit expression for the matrix K defined in eq. (3.6):

$$K = \frac{1}{\Lambda^T\Lambda}\begin{pmatrix} u\lambda_1^2 + \lambda_2^2 & \lambda_1\lambda_2(u-1) \\ \lambda_1\lambda_2(u-1) & \lambda_1^2 + u\lambda_2^2 \end{pmatrix}, \tag{4.1}$$

where $u = |\lambda_0|/(\lambda_0 + \lambda_1^2 + \lambda_2^2)^{1/2}$. Using this expression, we solve the constraint (3.3) in terms of the 2×2 matrix p. As there is only one explicit i in the last term of eq. (3.3), the hermitan matrix p should be pure imaginary and antisymmetric. All we need to compute is then the single coefficient of two-dimensional antisymmetric matrix. After some algebra and rearrangement, we find the matrix p. To express the answer in compact form, we introduce two cyclic quantities for two instantons,

$$\mathcal{P} = \frac{4\, q^a\eta^a_{\mu\nu}\,(a_{0\mu}a_{1\nu} + a_{1\mu}a_{2\nu} + a_{2\mu}a_{0\nu})}{(s_0s_1)^{-1}|a_0 - a_1|^2 + (s_1s_2)^{-1}|a_1 - a_2|^2 + (s_2s_0)^{-1}|a_2 - a_0|^2}\,, \tag{4.2}$$

and

$$\mathcal{F}(x) = \sigma^a\bar{\eta}^a_{\mu\nu}\left(\frac{y_{0\mu}}{|y_0|^2}\frac{y_{1\nu}}{|y_1|^2} + \frac{y_{1\mu}}{|y_1|^2}\frac{y_{2\nu}}{|y_2|^2} + \frac{y_{2\mu}}{|y_2|^2}\frac{y_{0\nu}}{|y_0|^2}\right). \tag{4.3}$$

The quantity $Q(x)$ of eq. (3.10) is given as $Q(x) = \mathcal{P}\mathcal{F}(x)$ which has the permutation symmetry between $i = 0, 1, 2$ indices. The resulting scalar field is then

$$\phi(x) = \frac{1}{s_\Sigma H(x)}(\bar{Z}qZ + \mathcal{P}\mathcal{F})\,, \tag{4.4}$$

JHEP09(2003)035

where $Z(x) = (s_0 y_{0\mu}/y_0^2 + s_1 y_{1\mu}/y_1^2 + s_2 y_{2\mu}/y_2^2)e_\mu$. We have also checked explicitly that the above solution satisfies the covariant Laplace equation. The relative sign between two terms in the numerator is crucial for the supertube configuration to appear. The solution (4.4) is our key result.

With this solution, we find the electric charge energy becomes

$$Q_e = \frac{8\pi^2}{s_\Sigma^2}\left\{(q^a)^2\sum_i s_i s_{i+1}(a_i - a_{i+1})^2 - \frac{4(\sum_i q^a \eta^a_{\mu\nu} a_{i\mu} a_{(i+1)\nu})^2}{\sum_j (s_j s_{j+1})^{-1}(a_j - a_{j+1})^2}\right\}, \tag{4.5}$$

where $q = q^a \sigma^a$ and the sum is over $0, 1, 2$ indices with the identification $a_3 = a_0$, $s_3 = a_0$. After putting three points a_0, a_1, a_2 on the 3-4 plane and using the Schwartz inequality and general property of triangles, one can easily show that

$$Q_e \geq \frac{16\pi^2}{s_\Sigma^2}\frac{(q^a)^2(\sum_i (a_i^2)^2)}{\sum_i (s_i s_{i+1})^{-1}(a_i - a_{i+1})^2}\,. \tag{4.6}$$

This shows that Q_e is positive, as it should be.

Let us now consider the zeros of the Higgs field. Without loss of generality, one can rotate the coordinate and put three instanton positions a_i on the 3-4 plane. When $q^3 = 0$, we note that $\mathcal{P}$ vanishes. In this case, the zeros of ϕ are determined by the zeros of $Z = Z_\mu e_\mu$ as H is positive definite and $1/H$ does not vanish anywhere. Due to the symmetry, the zeros of Z should be also on 3-4 plane. Then Z vanishes at points where the sum of two dimensional 'Coulomb forces' (due to three positive 'charges' s_0, s_1, s_2 at a_0, a_1, a_2) vanishes. The zero points are critical points of the potential. Generically there are two such points. Consider two charges close to each other and one far apart. The equipotential lines surrounding two close charges merge together, generating one such zero point. Then this merged equipotential lines will merge with those from the third charge, generating another zeros. Later we will write down the algebraic equation for the zero points which shows that there are two zero points indeed. On the other hand when $q^3 \neq 0$ and $q^1, q^2 = 0$, $\mathcal{P}$ does not vanish. Again in this case, symmetry implies that the zeros of the scalar field should be confined to the 3-4 plane. We will see they form a curve on the 3-4 plane. If all components of q^a do not vanish, the zeros of the Higgs field do not remain on the 3-4 plane and would become far more complicated, so we will not pursue this case here.

4.1 A derivation of the structure of porism

The JNR two-instanton solution carries 15 parameters (3 four-positions and 3 scales), two more than 13 moduli as we expect for two-instanton solution. The overall scale is trivial as it leaves the potential (2.7) invariant. Another unphysical parameter is related to a local gauge transformation. First put three points $a_{0\mu}, a_{1\mu}, a_{2\mu}$ on a circle on the 3-4 plane. The lines connecting them forms a triangle inside the circle. It is known that, under certain local gauge transformation, the three points a_0, a_1, a_2 move along the circle with speeds proportional to s_0, s_1, s_2, respectively. There exists an ellipse inside the circle which touches all sides of the one-paramter family of triangles tangentially. This ellipse also remains invariant under the local gauge transformation. This fact also determines how the scale parameters should transform under the local gauge transformation. This one parameter family of triangles is the so-called porism of triangles [14]. However, the physical meaning of this ellipse remains obscure.

It seems that the already-known derivation of this result is somewhat involved. Here we describe another way to derive the porism by using our Higgs solution (4.4). As the scalar field transforms homogeneously under local gauge transformations, the zero points of the scalar field are gauge invariant. Especially with $a_{i\mu}$ on the 3-4 plane and $q^3 = 0$, the scalar field has at most two isolated zero points on the 3-4 plane, which should be invariant under local gauge transformations. We aim to derive the deformation rule by studying a single equation $Z(x) = 0$.

For our $\kappa = 2$ case, $Z(x) = 0$ amounts to a quadratic equation of $x_\mu e_\mu$ on the 4-3 plane. We put three points on a circle whose radius is R and whose center is at the origin. In complex notation, we put $a_{j4} + ia_{j3} = Re^{i\theta_j}$ and a point $z = x_4 + ix_3$ on the 3-4 plane. The zeros of $Z(x)$ satisfy a complex equation

$$z^2 + C_1 Rz + C_2 R^2 = 0\,, \tag{4.7}$$

where

$$C_1 = -\sum_{j=0}^{2}\left(1 - \frac{s_j}{s_\Sigma}\right)e^{i\theta_j}, \quad C_2 = e^{i(\theta_0+\theta_1+\theta_2)}\sum_j \frac{s_j}{s_\Sigma}e^{-i\theta_j}\,. \tag{4.8}$$

As the above equation is quadratic, there can be at most two isolated zeros. They are determined by two complex numbers C_1, C_2. We are looking for a change of parameters which leave the circle and the Higgs zeros invariant. That is an arbitrary variation of the angles θ_i and s_i which leaves s_Σ, C_1, C_2 invariant. As there are five real paramters in s_Σ, C_1, C_2 to fix and six parameters θ_i and s_i to vary, there is one free parameter among θ_i and s_i, which is the porism of triangles.

Under infinitesimal change $\delta\lambda_i$ and δs_i such that $\delta s_\Sigma = 0$, we note that

$$\delta C_1 - e^{i\sum_i \theta_i}\delta\bar{C}_0 = i\sum_{j,k} e^{i\theta_j}\left(-s_j\delta\theta_k + s_j\delta\theta_k\right). \tag{4.9}$$

The vanising condition of the above term for generic s_j, θ_j implies

$$\delta\theta_j = s_j\delta t \tag{4.10}$$

with an infinitesimal parameter t. As

$$\delta C_1 = \sum_j e^{i\theta_j}\left(\frac{\delta s_j}{s_\Sigma} - i\left(1 - \frac{s_j}{s_\Sigma}\right)\delta\theta_j\right), \tag{4.11}$$

the equation $\delta C_1 = 0$ implies a complex equation

$$\sum_j e^{i\theta_j}\left(\frac{\delta s_j}{s_\Sigma} - i\left(1 - \frac{s_j}{s_\Sigma}\right)s_j\delta t\right) = 0 \tag{4.12}$$

for two independent δs_j. Therefore, the structure of porism is completely derived from gauge-invariance of the zero points of the Higgs field.

JHEP09(2003)035

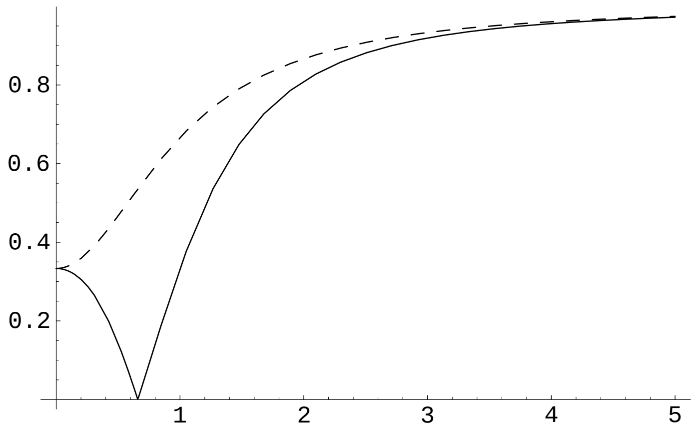

Figure 1: The Higgs field profile in the unit of $q^3\sigma^3$: the solid line is along x^3 axis and the dashed line is along x^1 axis.

4.2 Supertube connecting D4-branes

Let us now consider the zeros of the scalar field when $q^3 \neq 0$ and $q^1, q^2 = 0$ with a_i on the 3-4 plane. The zeros of the scalar field would lie on the 3-4 plane due to the symmetry and is given by the solutions of a real equation

$$X = \sum_{i=0}^{2} (s_i^2\, y_{i+1}^2 y_{i+2}^2 + 2 s_i s_{i+1}\, y_i \cdot y_{i+1}\, y_{i+2}^2 - \mathcal{P}' y_{i+2}^2\, y_i \times y_{i+1}) = 0 \tag{4.13}$$

for a two dimensional vector $x = (x_3, x_4)$, where $y_i = x - a_i$ become two dimensional vectors on the 3-4 plane and

$$\mathcal{P}' = \frac{4\,(a_{0\mu} \times a_{1\nu} + a_{1\mu} \times a_{2\nu} + a_{2\mu} \times a_{0\nu})}{(s_0 s_1)^{-1}|a_0 - a_1|^2 + (s_1 s_2)^{-1}|a_1 - a_2|^2 + (s_2 s_0)^{-1}|a_2 - a_0|^2}\,. \tag{4.14}$$

Here the cross product is the two dimensional skew product, $a \times b = a_3 b_4 - a_4 b_3$.

Let us consider a simple case where three points a_i are the vertices of an equitriangle, $a_0 = (-1, 0)R$, $a_1 = (1/2, -\sqrt{3}/2)R$, and $a_2 = (1/2, \sqrt{3}/2)R$ on the 3-4 plane, and all scales are such that $s_i = 1$. Then

$$X = 9(x^2)^2 + \left(6 - \frac{3\sqrt{3}}{2}\mathcal{P}'\right) R^2 x^2 - \frac{3\sqrt{3}}{2}\mathcal{P}' R^4 = 0\,. \tag{4.15}$$

As $\mathcal{P}' = 2/\sqrt{3}$ in this case, the above equation has zero at points

$$x^2 = \frac{\sqrt{13} - 1}{6} R^2 \tag{4.16}$$

which is a square of the radius of a circle. Since the scalar field has nontrivial expectation value at spatial infinity and has zeros on a circle, two D4-branes separated along the transverse direction come together on a circle. There is no special degeneracy of zeros and so the circle becomes the magnetic monopole string. We interprete this dyonic configuration as the supertube inserted between two separated D4-branes. The value of the Higgs field at the center of the circle or the orgin is $q^3\sigma^3/3$, which is smaller than the asymptotic value, regardless the size of the equitriangle. As the size R increases, the electric charge increases and is able to deform the Higgs field so that the value at the center remains smaller. The value of the Higgs field along x^1 axis and x^3 axis with $R = 1$ are drawn in figure 1.

Let us now consider a little more general case. Instead we choose $a_0 = (-s_0, 0)$, $a_1 = (1/2, -\sqrt{3}/2)$, and $a_2 = (1/2, \sqrt{3}/2)$ on the 3-4 plane with $R = 1$. We choose s_0 arbitrary and $s_1 = s_2 = 1$. Then we change the scale s_0 from 1 to ∞ gradually, which interpolates between the JNR and the 't Hooft solutions of two instantons. (When we change s_0 from 1 to 0, the solutions interpolate between the JNR and the 't Hooft solution of a single instanton. The zeros of the Higgs changes acoordingly.) In the 't Hooft case, the zeros are located at two points a_1, a_2. In figure 2, we draw the zeros of the Higgs field for the various values of s_0. For the $s_0 = 1$, the zeros make a circle as we described before. The figure shows the gradual deformation of the circle, or the magnetic monopole string, into two points. (The curve of the zeros is not convex and is not clearly related to the ellipse of the porism.) This illustrates that the 't Hooft dyonic instanton indeed represents collapsed tubes.

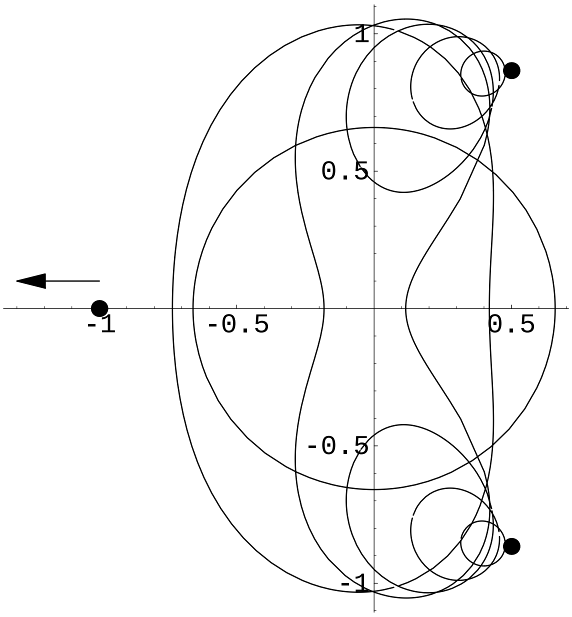

Figure 2: Deformation of Magnetic monopole strings for various values of s_0.

5. Conclusion

We investigated dyonic two instanton configurations with the JNR ansatz, which is most general two instanton solution. We have shown that the monopole string structure appears with this solution, which leads naturally to the interpretation of dyonic instantons as supertubes between D4-branes. We have exlored the detail of the porism of triangle in the JNR ansatz and have also seen how the monopole string deforms as the moduli parameters change, especially as the JNR solution deforms to 'tHooft solution of two instantons.

It would be natural to look for how our construction generalizes to higher instanton number case. At least with the JNR ansatz, it seems to be doable to find the explicit form of the Higgs field with some effort. Our preliminary investigation shows that there is a hierarchy of clustering of terms in the Higgs solution.

By the way, our solution (4.4) might be useful in a different direction. To find the moduli space of instantons, one has to find infinitesimal variation of the gauge field which satisfies the covariant gauge or the Gauss law. As far as we know, no explicit moduli space metric has been found for two instanton solutions. The adjoint Higgs field we found is intimately related to the global gauge zero modes of instantons. The electric energy would be the kinetic energy for the global gauge transformation with the gauge parameters given by q^a. It would be a challenge to find the whole moduli space metric for two instantons.

Acknowledgments

The authors wish to thank Dongsu Bak and Piljin Yi for the enlightening discussions. S.K. is supported in part by the BK21 project of the Ministry of Education, Korea, and the Korea Research Foundation Grant 2001-015-DP0085. K.L. is supported in part by KOSEF 1998 Interdisciplinary Research Grant.

References

[1] N.D. Lambert and D. Tong, *Dyonic instantons in five-dimensional gauge theories*, *Phys. Lett.* **B 462** (1999) 89 [hep-th/9907014].

[2] M. Zamaklar, *Geometry of the nonabelian DBI dyonic instanton*, *Phys. Lett.* **B 493** (2000) 411 [hep-th/0006090].

[3] E. Eyras, P.K. Townsend and M. Zamaklar, *The heterotic dyonic instanton*, *J. High Energy Phys.* **05** (2001) 046 [hep-th/0012016].

[4] K. Peeters and M. Zamaklar, *Motion on moduli spaces with potentials*, *J. High Energy Phys.* **12** (2001) 032 [hep-th/0107164].

[5] K.-M. Lee and S.-H. Yi, *1/4 BPS dyonic calorons*, *Phys. Rev.* **D 67** (2003) 025012 [hep-th/0205274].

[6] D. Mateos and P.K. Townsend, *Supertubes*, *Phys. Rev. Lett.* **87** (2001) 011602 [hep-th/0103030].

[7] D. Mateos, S. Ng and P.K. Townsend, *Tachyons, supertubes and brane/anti-brane systems*, *J. High Energy Phys.* **03** (2002) 016 [hep-th/0112054].

[8] M. Kruczenski, R.C. Myers, A.W. Peet and D.J. Winters, *Aspects of supertubes*, *J. High Energy Phys.* **05** (2002) 017 [hep-th/0204103].

[9] D. Bak and K. Lee, *Supertubes connecting D4-branes*, *Phys. Lett.* **B 544** (2002) 329 [hep-th/0206185].

[10] D. Bak and K.-M. Lee, *Noncommutative supersymmetric tubes*, *Phys. Lett.* **B 509** (2001) 168 [hep-th/0103148].

[11] D. Bak and A. Karch, *Supersymmetric brane-antibrane configurations*, *Nucl. Phys.* **B 626** (2002) 165 [hep-th/0110039].

[12] R. Emparan, D. Mateos and P.K. Townsend, *Supergravity supertubes*, *J. High Energy Phys.* **07** (2001) 011 [hep-th/0106012].

[13] R. Jackiw, C. Nohl and C. Rebbi, *Conformal properties of pseudoparticle configurations*, *Phys. Rev.* **D 15** (1977) 1642.

[14] M.F. Atiyah and N.S. Manton, *Geometry and kinematics of two skyrmions*, *Commun. Math. Phys.* **153** (1993) 391.

[15] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of instantons*, *Phys. Lett.* **A 65** (1978) 185.

[16] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *The calculus of many instantons*, *Phys. Rept.* **371** (2002) 231 [hep-th/0206063].

[17] N.H. Christ, E.J. Weinberg and N.K. Stanton, *General self-dual Yang-Mills solutions*, *Phys. Rev.* **D 18** (1978) 2013.